多変数の微分積分学２ 問12解説

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2007年12月13日, 12月20日

問12. C^2 級のベクトル場 $\boldsymbol{f}\colon \mathbf{R}^3 \to \mathbf{R}^3$ に対して、次の (1), (2) が成り立つことを示せ。
(1) $\operatorname{div}(\operatorname{rot}\boldsymbol{f}) = 0$　(2) $\operatorname{rot}(\operatorname{rot}\boldsymbol{f}) = \operatorname{grad}(\operatorname{div}\boldsymbol{f}) - \triangle\boldsymbol{f}$

(1)

$$\operatorname{rot}\boldsymbol{f} = \det\begin{pmatrix} \dfrac{\partial}{\partial x_1} & f_1 & \boldsymbol{e}_1 \\ \dfrac{\partial}{\partial x_2} & f_2 & \boldsymbol{e}_2 \\ \dfrac{\partial}{\partial x_3} & f_3 & \boldsymbol{e}_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_3}{\partial x_2} - \dfrac{\partial f_2}{\partial x_3} \\ \dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_3}{\partial x_1} \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2} \end{pmatrix}.$$

f_i が C^2 級なので 2 階偏導関数は偏微分の順序に依らないことから、

$$\begin{aligned}\det(\operatorname{rot}\boldsymbol{f}) &= \frac{\partial}{\partial x_1}\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \\ &= \frac{\partial^2 f_3}{\partial x_1 \partial x_2} - \frac{\partial^2 f_2}{\partial x_1 \partial x_3} + \frac{\partial^2 f_1}{\partial x_2 \partial x_3} - \frac{\partial^2 f_3}{\partial x_2 \partial x_1} + \frac{\partial^2 f_2}{\partial x_3 \partial x_1} - \frac{\partial^2 f_1}{\partial x_3 \partial x_2} = 0.\end{aligned}$$